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On the Application of Fourier's Double Integrals to Optical Problems

Charles Godfrey

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X. *On the Application of Fourier's Double Integrals to Optical Problems.*

By CHARLES GODFREY, *B.A., Scholar of Trinity, Isaac Newton Student in the University of Cambridge.*

Communicated by Professor J. J. THOMSON, *F.R.S.*

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INTRODUCTORY.

§ 1. THE object of the following work is to make some progress with the mathematical representation of the motions which go to compose natural light.

§ 2. It has always been recognised that interference phenomena forbid us to regard any natural radiation as consisting of an unending train of simple waves, such as may be represented by sine functions. At the same time, the equations of optics find their simplest solution in circular functions. It is desirable to enquire how far we may resolve a natural luminous motion with a sum of simple wave-trains by means of FOURIER'S "Theorem of Double Integrals." This procedure was first suggested by GOUY.*

§ 3. Doubts have often been entertained as to the permissibility of this process. Writers have been sceptical as to the physical meaning and independence of the simple waves thus introduced. In the following pages will be found an attempt at a strict justification of the method. It is based upon two principles (i.) that we are cognisant of light only by means of the integral effects produced by the light during an interval of time which depends upon the nature of the detector in use (the eye, a photographic plate, &c.); (ii.) that we are not concerned with simple wave-lengths, but rather with short ranges of wave-length, whose integrated energy we observe. The former principle is generally accepted; the latter has been put forward with great force by Lord RAYLEIGH.†

§ 4. In what follows we shall deal solely with plane and plane-polarised light.

§ 5. The matter at issue cannot be introduced better than by a quotation from GOUY‡:—

"On sait que la théorie ondulatoire, dans les explications qu'elle donne des phénomènes optiques, a pour objet immédiat le *mouvement simple*, dans lequel la vitesse vibratoire‖ d'un point quelconque est donnée par une équation de la forme

$$\nu = a \sin 2\pi\left(\frac{t}{\theta} + b\right)$$

* GOUY, 'Journn. de Physique,' ser. 2, vol. 5, p. 354. (1886.)

† Lord RAYLEIGH, 'Phil. Mag.,' vol. 27, 1889.

‡ GOUY, 'J. de Ph.,' ser. 2, vol. 5, p. 354.

‖ It is clearly immaterial whether we speak of velocities and displacements of an elastic medium, or of electric and magnetic forces. The real objects of discussion are vectors, which can be interpreted in various ways.

t désignant le temps, a, b, et θ des constants. Cette équation définit une suite entièrement *illimitée* de vibrations pendulaires, d'une *régularité absolue*, dont la période est θ.

"Si ces conditions de regularité et de durée ne se trouvent pas rigoureusement réalisées, l'équation du mouvement est différente, et, par suite, un problème nouveau se trouve posé, la solution fournie par la théorie pour un mouvement simple n'étant plus applicable en général. S'il s'agit, par exemple, d'un phénomène d'interférence ou de diffraction, on voit immédiatement que toute irrégularité et toute interruption entraîne une perturbation dans l'effet produit, comme on l'a remarqué depuis longtemps. Il en sera encore de même, abstraction faite des interférences, toutes les fois que l'on aura à considérer des milieux doués de dispersion. En effet, le mouvement vibratoire dans les divers milieux devant toujours satisfaire aux équations différentielles des petits mouvements de ces milieux, il n'est nullement permis de faire subir une altération, quelle qu'elle soit, au mouvement vibratoire (i.) et de supposer en suite que ce mouvement se comportera, dans les milieux doués de dispersion, comme s'il n'était pas altéré. Ainsi, par exemple, on n'est pas en droit de supposer que le mouvement (i.) ne comprend qu'un nombre de vibrations limité, et qu'il se propagera dans les divers milieux comme s'il formait la suite entièrement illimitée que definit l'équation (i.).

"D'autre part, on a remarqué depuis longtemps qu'aucune source lumineuse ne peut produire une série de vibrations indéfinie et parfaitement régulière, ne fût-ce qu'en raison du renouvellement incessant des particules incandescentes. Ainsi aucun mouvement lumineux réel, même le moins complexe qu'on puisse supposer, ne rentre rigoureusement dans le cas du mouvement simple que considère la théorie ondulatoire.

"Cette difficulté est présentée dès l'origine de cette théorie. On y répond d'ordinaire en supposant que les sources lumineuses produisent des séries de vibrations régulières, mais troublées de temps à autre par des perturbations subites ou de courte durée. Si la série, entre deux perturbations, comprend un grand nombre de vibrations, on peut prévoir que l'effet moyen d'un pareil mouvement différera peu de celui d'un mouvement simple. Mais cet aperçu, reposant sur une hypothèse, ne peut servir de base à une étude rationnelle du problème qui nous occupe, et nous verrons que, pour les sources donnant des spectres continus, on peut se faire une idée beaucoup moins étroite au mouvement lumineux. De plus, cet aperçu ne nous apprend rien sur les effets des perturbations elles-mêmes, qui paraissent jouer un rôle important dans la constitution des spectres fournis par les vapeurs et les gaz incandescents."

§ 6. The general process to which GOUY alludes is the analysis of any disturbance whatever by means of FOURIER'S theorem. He considers a function which is defined within a given interval of time; this is analysed into a sum of circular functions of time; the periods of the terms being the interval itself and all sub-multiples of it.

It seems unnecessary to restrict the function by defining it for a finite interval alone. We may at once discuss a motion given for all values of the time, from $-\infty$ to $+\infty$.

By the theorem of FOURIER's double integrals*

$$f(t) = \int_0^\infty (\mathrm{C}\cos ut + \mathrm{S}\sin ut)du$$

where $$\mathrm{C} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos uvdu, \quad \mathrm{S} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin uvdv.$$

The disturbance is thus analysed into a sum of elementary simple vibrations, of which

$$du(\mathrm{C}\cos ut + \mathrm{S}\sin ut)$$

is typical.

Each of these is a simple circular function; the results of the undulatory theory are directly applicable to it.

The periods of the elementary vibrations have all values from zero to infinity. Now, if the disturbance could be analysed as a number of simple circular functions with distinct periods, the separate elements would have meaning, as in the familiar harmonic analyses of tides, vibrations of musical instruments, &c. The question we have to answer is, have the simple elements meaning in the limit, when their number is infinite, and the sum becomes an integral?

§ 7. We will at once notice an obvious criticism; this was, in fact, offered by POINCARÉ,† soon after GOUY's article appeared. Each of the component vibrations $du(\mathrm{C}\cos ut + \mathrm{S}\sin ut)$ exists unchanged through all time. This is true whatever be the nature of the disturbance we are analysing. But this disturbance may, for instance, be zero, except within a certain definite interval of time. Take the case of a flash of light. Now a spectroscope, says M. POINCARÉ, will separate the component vibrations laterally; they may be examined separately. Hence a spectroscope will enable us to see the light for an infinite time before it is kindled, and for an infinite time after it is extinguished. The analysis must therefore be fallacious.

The answer to this objection is as follows. No spectroscope possesses infinite analysing power. A given point at the focus of the telescope will be illuminated by light of a whole range of periods. Or, to look at the matter from another point of

* *On the Applicability of Fourier's Double Integral to Functions occurring in Physical Problems.*—In pure mathematics the applicability of FOURIER's theorems to functions is subject to certain limitations. These limitations exist when the functions possess infinite sets of discontinuities, or infinite sets of fluctuations, or infinities of certain types. Now, in concrete physical cases, we find neither infinities nor discontinuities. It is true that infinities and discontinuities may occur in functions commonly used to represent physical quantities. But the presence of such features is due to the abstract character of the method; a function more closely realising the properties of the physical quantity in question would be without infinite or discontinuous features.

† POINCARÉ, "Spectres Cannelés," 'C. R.,' 120, pp. 757–762, 1895; also see SCHUSTER, 'C. R.,' 120, pp. 987–989, 1895.

view, a perfectly monochromatic train of waves will, by virtue of diffraction, illuminate, not a point, but a small area of the focal plane. The different elements of the Fourier integral will not be distinguished separately; they will to some extent be superposed and recombine. The result will be, at each point of the focal plane, a disturbance not altogether different from the original motion in duration and character. We see then that the Fourier analysis may after all have meaning and application, and not lead to a paradox such as POINCARÉ anticipated.

It must be noticed that this recombination of the different elements of the integral is essentially connected with the phase-relation which exists between the said simple elements.

§ 8. We have seen that POINCARÉ'S objection will not prevent us from regarding the original ether-motion as mathematically equivalent to the Fourier integral. Whatever services the Fourier analysis can render us we may safely accept.

It will be found that the different simple elements of the Fourier integral cannot *in general* be said to have any independent physical existence. On the other hand, part of the following essay is an attempt to prove that in certain cases the different Fourier elements can be regarded as having such physical existence. A special case of this nature is that of a steady emission, such as the radiation of an incandescent gas. We shall inquire to what extent such radiations are equivalent to mixed light, presenting a continuous spectrum of composition determined by Fourier analysis.

The Fundamental Theorem.

§ 9. We will now introduce a theorem proved by Professor SCHUSTER.* A particular case was given by Lord RAYLEIGH.†

The theorem is as follows :—

$$\int_{-\infty}^{\infty} f(t)\phi(t)dt = \frac{1}{\pi}\int_0^{\infty}(A_1A_2 + B_1B_2)du,$$

where

$$A_1 = \int_{-\infty}^{+\infty} f(\lambda)\cos u\lambda d\lambda, \qquad B_1 = \int_{-\infty}^{+\infty} f(\lambda)\sin u\lambda d\lambda$$

$$A_2 = \int_{-\infty}^{+\infty} \phi(\lambda)\cos u\lambda d\lambda, \qquad B_2 = \int_{-\infty}^{+\infty} \phi(\lambda)\sin u\lambda d\lambda.$$

In other words, if

$$f(t) = \int_0^{\infty} R_1 \cos(ut + \psi_1)du$$

$$\psi(t) = \int_0^{\infty} R_2 \cos(ut + \psi_2)du$$

* Professor SCHUSTER, 'Phil. Mag.,' vol. 37, p. 533, 1894.

† Lord RAYLEIGH, 'Phil Mag.,' vol. 27, 1889.

where R_1, R_2, ψ_1, ψ_2 are functions of u,

then
$$\int_{-\infty}^{+\infty} f(t)\phi(t)dt = \pi\int_0^{\infty} R_1R_2 \cos(\psi_1 - \psi_2)du.$$

The integrated value of $f(t)\phi(t)$ depends therefore upon the distribution of energy in the separate elements, and upon the *difference of phases* of corresponding elements in the integrals.

§ 10. Professor SCHUSTER needs the theorem in order to prove that, in case of a bifurcated beam of light interfering with itself, "the amount of interference depends on the distribution of energy only, and not on any assumption respecting the regularity or irregularity of vibration."

The proposition has, however, very much wider consequences.

§ 11. For the present, the discussion will be confined to the particular case of *constant light*, *i.e.* light which does not present any *perceptible* fluctuations or other time-features.

§ 12. All our cognizance of radiation other than the long waves of HERTZ is by average effects. We average over a length of time great compared with the periods of vibration. This is true whatever be the means used to perceive and register the radiations, whether by direct visual perception, or by chemical effect, photographic or other, or by heating effect (bolometric), or by luminescence which the radiation excites. For Hertzian waves, on the other hand, the features of a single wave can be discovered.

In discussing the qualitative effect of constant light, with a view to discriminating between different wave-lengths, we are concerned solely with the integral effect over a certain interval of time.

It is doubtful how far we are at liberty to consider the molecule as a simple vibrator. But, in so far as this assumption is justified, we may prove that the observed effects of constant light will depend on nothing but the partition of energy among the different elements of the equivalent Fourier integral. The phases will of course determine whether the light shall be "constant" or no; but, if that condition is fulfilled, their further influence will not be perceived.

§ 13. *Energy.*—The whole energy of the light motion $f(t)$ depends upon $\int_{-\infty}^{+\infty} f^2(t)dt$

By the theorem of SCHUSTER and RAYLEIGH, this is equal to
$$\pi\int_0^{\infty} R^2 du$$

where
$$f(t) = \int_0^{\infty} R\cos(ut + \psi)du.$$

Interference with given path difference 2T.

This depends on

$$\int dt[\{f(t)+f(t+2\tau)\}^2 - f^2(t) - f^2(t+2\tau)] = 2\int f(t)f(t+2\tau)dt.$$

Now $$f(t) = \int_0^\infty \mathrm{R}\cos(ut+\psi)du$$

$$f(t+2\tau) = \int_0^\infty \mathrm{R}\cos(ut+2u\tau+\psi)du,$$

$$\therefore \qquad \int_{-\infty}^{+\infty} f(t)f(t+2\tau)dt = \pi\int_0^\infty \mathrm{R}^2\cos 2u\tau du.$$

The phase ψ has disappeared.*

Influence of Light on a Vibrator.

The phenomena of refraction, dispersion and absorption, can be explained (subject to the above reservation) by considering the action of light waves on a vibrator.

The equation of motion of the vibrator is

$$\ddot{x} + 2k\dot{x} + p^2x = f(t) = \int_0^\infty \mathrm{R}\cos(ut+\psi)du$$

$$= \text{real part of} \int_0^\infty \mathrm{R}e^{i(ut+\psi)}du.$$

The "general" solution of this equation will be of the form $\mathrm{A}e^{-kt}\cos(rt+\phi)$, where $r^2 = p^2 - k^2$ and A, ϕ are arbitrary. The above solution is to hold for all time from $-\infty$ to $+\infty$. We must therefore put $\mathrm{A} = 0$, and the complete solution will be

$$x = \text{real part of} \int_0^\infty \frac{\mathrm{R}e^{i(ut+\psi)}du}{p^2 - u^2 + 2kiu}\dagger$$

$$= \int_0^\infty \frac{\mathrm{R}du}{(p^2-u^2)^2 + 4k^2u^2}\{(p^2-u^2)\cos(ut+\psi) + 2ku\sin(ut+\psi)\}.$$

The average energy of the motion excited will depend upon

$$\int_0^\infty \frac{\mathrm{R}^2du}{(p^2-u^2)+4k^2u^2}.$$

Again, the work done by the light depends upon $\int f(t)\dot{x}dt$. Applying SCHUSTER'S

* SCHUSTER, 'Phil. Mag.,' vol. 37, p. 533.

† We verify that this is a solution. This involves the process of differentiating inside the integral. Now, the condition that a Fourier integral shall admit of being so treated is, that the function represented shall be free from discontinuities and shall vanish at $\pm\infty$. No mathematical discontinuities will occur in a physical problem; and, if necessary, the conditions at infinity may be satisfied by introducing into $f(t)$ a factor (such as $e^{-a^2t^2}$ where a is small) which shall ensure dying away at both extremities, and which at the same time will not affect the Fourier resolution in any marked degree.

theorem, we find that the terms connecting different periods drop out (a familiar property of ordinary harmonic analysis), and the *rate of absorption* is dependent on

$$\pi\int_0^\infty \frac{2ku^2R^2du}{(p^2-u^2)^2+4k^2u^2}$$

The phase ψ has again disappeared.

Heating effects are directly dependent upon absorption. So again with *physiological effects.*

§ 14. In the case of *chemical* and *electrical* effects produced by light we probably have some kind of dissociation. This is perhaps true of *luminescence* also. It may be that we do not yet understand the mechanism of dissociation. But, if the dissociation arises from separation of ions as their light-excited vibrations become large, the vibrator analogy will apply here as well. Doubtless some molecules will split up sooner and others later; for the individual molecule the precise timing of its own vibrations with the phase of the incident light will be all-important. But on the average of a large number of molecules, the amount of dissociation will perhaps depend on the rate of absorption of energy by a vibrator typifying the average structure.

It is necessary to repeat that our assumption of a linear equation for the vibration of a molecule cannot be regarded as more than a first step towards a solution of a difficult problem. In the words of Sir GEORGE STOKES, "Linearity applies to the small disturbance of the single elastic medium—the ether—but it does not follow that linearity applies to all the effects produced in a complex system of molecules."

§ 15. Let us consider the application of the present treatment to the *spectroscopic analysis* of light.

The light emergent from the instrument in a given direction is compounded of different wave-lengths. The element

$$R\cos(ut+\psi)du$$

of the integral will contribute a component

$$R\phi(u)\cos(ut+\psi)du.$$

In this expression $\phi(u)$ depends upon the structure of the instrument, and the direction chosen, as well as upon u. The change of phase $\psi-\theta$ depends upon the same causes. We must note, however, that neither $\phi(u)$ nor $\psi-\theta$ depends upon ψ.

The emergent light will be

$$\int_0^\infty R\phi(u)\cos(ut+\theta)du.$$

Now we have seen that the phase only enters as determining the constancy of the light. If the light which comes into the instrument is constant, so also is the emergent beam. The phase has no further part to play; hence, the *spectroscopic* *analysis of constant light depends solely upon the instrument and the distribution of energy among the elements of the Fourier integral.*

§ 16. A dispersive medium, apart from its possible selective absorption of the different wave-lengths, will always alter the relative phases of the different elements. The transmitted light will thus be altered. But the preceding work has shown that these phase-changes will not affect the sensible properties of the light.

§ 17. We have arrived at the conclusion that the different simple components of constant light are not only superposed, but also independent as regards all energy properties.

Radiations composed of a random Aggregate of Pulses.

§ 18. A constantly-recurring problem in optics is that of the composition of an irregular sequence of pulses of a given type.

The question occurs in dealing with the radiation of an incandescent gas. The pulse here consists of the train of waves given off by the molecule during its free path; after an encounter the train will be entirely changed, and practically independent of the former train.

Again, what is the total effect on radiation of the damping to which the vibrations of the molecules are subject? The question was raised by LOMMEL.* This author was content to analyse $e^{-kt} \sin (pt + \psi)$ as a Fourier integral, and assume that the different elements are independent. This, of course, will not be true for the simple pulse which LOMMEL considered. It is true that the motion $e^{-kt} \sin (pt + \psi)$ can be reconstructed by means of an infinite series of vibrators whose amplitudes follow the law of the Fourier expansion. But the phases of these vibrators will not be independent; they must be carefully adjusted to give the requisite effect.

§ 19. We shall find that, when we deal with an infinite and irregular succession of such pulses, the energy properties do completely specify the motion. The disturbing influence of phase will disappear; in the Fourier integral representing the complete motion, the phase will be a rapidly-fluctuating function of the wave-length, and all distinctive phase-properties will average out.

The omission to deal with a sequence of pulses has exposed LOMMEL's analysis to adverse criticism. It will be seen that a more complete treatment confirms the results which he obtained as regards the widening of spectrum lines through damping.

§ 20. Another case in point is that of Röntgen rays. These are satisfactorily covered by Professor J. J. THOMSON's theory of electric pulses. The pulses are of given type; each one may be analysed by FOURIER's theorem. We find a certain energy-wave-length curve; in dealing with an infinite succession of pulses the phase-

* LOMMEL, 'Wied. Ann.,' 3, 251, 1878.

relations disappear, and we are left with the energy curve to completely specify the properties of the sequence of pulses. These statements will be justified below.

§ 21. The subject has been opened by Lord RAYLEIGH* in his paper on "The Complete Radiation at a given Temperature." He proposes to regard this as an irregular sequence of pulses of the type $e^{-c^2x^2}$.

Now

$$e^{-c^2x^2} = \frac{1}{c\pi^{\frac{1}{2}}}\int_0^\infty e^{-u^2/4c^2}\cos ux du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and the whole energy of the pulse

$$\int_{-\infty}^{+\infty} e^{-c^2x^2}dx = \frac{1}{c^2}\int_0^\infty e^{-u^2/2c^2}du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The intensity corresponding to the limits u and $u + du$ is therefore $c^{-2}e^{-u^2/2c^2}du$.

"If an infinite number of impulses, similar but not necessarily equal to (8), and of arbitrary sign, be distributed at random over the whole range from $-\infty$ to $+\infty$, the intensity of the resultant for an absolutely definite value of u would be indeterminate. Only the *probabilities* of various resultants could be assigned; and if the value of u were changed, by however little, the resultant would again be indeterminate. Within the smallest assignable range of u there is room for an infinite number of independent combinations. We are thus concerned only with an average, and the intensity of each component may be taken to be proportional to the total number of impulses (if equal) without regard to their phase-relations. In the aggregate vibrations, the law according to which the energy is distributed is still, for all practical purposes, that expressed by (22)."

§ 22. This important paragraph suggests the whole theory. But when we come to take a closer view of it, it will be found that there are certain questions which still remain to be solved.

Suppose, for instance, that the elementary pulse is confined to a certain small range of time, such as are the pulses in Professor THOMSON'S theory, how many of those pulses must be present in order to give the properties which Lord RAYLEIGH associates with an infinite succession? Again, we might suppose that different consequences would follow from different degrees of crowding among the pulses. They may be so close, on the average, that a great number of them are everywhere found overlapping; or, again, they may be so thinly scattered as to be, on the average, far apart in comparison with the space occupied by each. Experiment does not allow us to say which of these suppositions is correct; we may enquire what is the test which determines the applicability of Lord RAYLEIGH'S theorem to the aggregate.

§ 23. In order that the sequence of pulses may, for us, be equivalent to a spectrum,

* 'Phil. Mag.,' 27, 1889.

an obvious condition is that the pulses shall not be so far apart as to be separately distinguishable. Photography can fix 10^{-7} second; hence there must be many pulses in 10^{-7} second. This is a condition certainly fulfilled by Röntgen rays. The coarser the means of observation which we use, the more thinly may the pulses be scattered. The results which we are about to investigate may be true, for a certain radiation, in the present state of experimental science; but will cease to be true for that particular kind of radiation when our instrumental means shall have been so improved as to enable us to distinguish structure in that radiation.

It shall be shown that this is the only condition necessary in order that a random sequence of similar pulses may be equivalent to radiation of a spectral composition given by the analysis of a single pulse.

§ 24. We have already proved that we are concerned simply with an integral effect over a time T of the order of the shortest observable interval. If we are content to view the radiation with the eye, or to use a slow photographic plate, T may be taken as great as we please. If, on the other hand, we are investigating the radiation with the shortest possible exposure, T may be reduced as far as our experimental skill will allow.

Let us examine the Fourier composition of a numerous sequence of random similar pulses.

Suppose the pulse to be

$$f(t) = \int_0^\infty \phi(u) \cos (ut + \psi) du$$

the angle ψ being a definite function of u. We are to examine the Fourier integral, equivalent to

$$f(t - \tau_1) + f(t - \tau_2) + f(t - \tau_3) + \ldots + f(t - \tau_n)$$

where $\tau_1, \tau_2, \tau_3, \ldots \tau_n$ define a large number of points of time distributed at random in an interval T. The breadth of the pulse is to be small compared with T.

The resultant integral is

$$\int_0^\infty \phi(u) \{\cos \overline{ut - u\tau_1 + \psi} + \cos \overline{ut - u\tau_2 + \psi} \ldots + \cos \overline{ut - u\tau_n + \psi}\} du.$$

Consider the quantity

$$\phi(u) \{\cos \overline{ut + \psi - u\tau_1} + \cos \overline{ut + \psi - u\tau_2} + \ldots + \cos \overline{ut + \psi - u\tau_n}\} \quad . \quad . \quad \text{A.}$$

First, suppose that the time of vibration, $2\pi/u$, is small compared with T or $\tau_n - \tau_1$. Draw from an origin lines of length $\phi(u)$ making with the prime vector angles $\psi - u\tau_1$, $\psi - u\tau_2$, &c. Then the bounding lines of the angles $\psi - u\tau_1$, $\psi - u\tau_2$, . . .

$\psi - u\tau_n$ lie at random all round an origin. The phase of the resultant will be arbitrary, while the mean value of its modulus is

$$n^{\frac{1}{2}} \cdot \phi(u).*$$

Now this is the amplitude of the compound harmonic motion A.

If we pass to a frequency $u + du$, where $\text{T}du$ is small, the new phases will differ but slightly from the old; but if $\text{T}du$ is finite or great, the new phases will differ finitely from the old, and the resultant for $u + du$ will have no apparent connection as regards phase with the resultant for u.

Let us consider what happens when we observe this radiation. First, we can only observe the content of a certain interval of time, which we have taken to be T; we receive into our apparatus the total energy of all the pulses in T. Now SCHUSTER'S theorem expresses the connection between the total energy of a radiation, and its expression as a Fourier integral (see p. 335). If the Fourier expression for the resultant of the n pulses in the present case is

$$\int_0^\infty \text{R} \cos (ut + \theta) du,$$

the total energy is

$$\pi \int_0^\infty \text{R}^2 du.$$

We have just seen that R is not a definite function of u, but partakes of the random character of the sequence of pulses. At this point we make use of RAYLEIGH'S principle; that we are not concerned with particular wave-lengths, but rather with the average energy over small ranges of wave-length. Bearing in mind the average value of R, we see that, for practical purposes, we have a spectrum whose intensity of energy for period $2\pi/u$ is

$$n\phi^2(u).$$

§ 25. The fluctuations in the energy-wave-length curve will be less rapid as we descend to the longer waves of the spectrum. As we have just seen, when the time of vibration is small compared with T, the fluctuations are so crowded as to be indistinguishable; the eye, or sensitive plate, will take the meancurve $n\phi^2(u)$. But if the time of vibration is large compared with T the range of angle included in the set

$$\psi - u\tau_1, \ \psi - u\tau_2, \ . \ . \ . \ \psi - u\tau_n$$

will be but small; the resultant will possess a phase intermediate between the extreme values, and a modulus of almost $n\phi(u)$. As we pass continuously to quicker vibrations, the modulus will diminish from $n\phi(u)$ to zero, and so on; the divergences of the energy curve from $\phi^2(u)$ being no longer rapid but on a broad and theoretically distinguishable scale.

It will be found, however, that for short pulses the amount of energy in the slow

* Lord RAYLEIGH, 'Phil. Mag.,' Aug., 1880; or 'Theory of Sound,' ed. 1894, p. 40.

waves is inconsiderable. The prepotent part of the energy resides in those quicker waves for which the energy curve is of the normal form $n\phi^2(u)$. (Compare the magnetic pulses of Professor THOMSON, treated in the next chapter).

§ 26. To recapitulate, the pulses of the sequence will not be separately distinguishable; their effect depends upon the integral of energy over an interval of time comparable with T; the phase of the Fourier element will have no further effect; and all the observable properties of the sequence will be bound up with the energy function $\phi^2(u)$.

§ 27. Hitherto it has been assumed that the interval T comprises a large number n of complete pulses, these being for the moment supposed not to be of infinite breadth. In general the boundaries of the interval T will find themselves *in* a pulse; there will be a number of incomplete pulses near each end. But these are few compared with the whole number n; they will not perceptibly affect the aggregate.

The spectrum is independent of n as regards composition, if n is large. The total intensity, however, varies as n; thus a variation in the crowding of the pulses causes a corresponding variation of the brightness of the spectrum—a result which might have been expected.

§ 28. Let us consider how these results are affected when the individual pulses are of infinite breadth. Suppose that we examine the type which Lord RAYLEIGH suggested,

$$f(t) = e^{-c^2t^2}.$$

The displacement becomes comparatively small when the distance from the centre of the pulse is great compared with $1/c$. In fact, we are tempted to regard these pulses as practically equivalent to pulses of finite breadth $1/c$ or thereabouts.

Suppose that the least observable interval T comprises a large number of central points of pulses. Suppose also that T is great compared with $1/c$. The interval will contain a contribution from each of the infinite succession of pulses. But, since $\int_b^\infty e^{-x^2}dx$ is small when b is great, only those pulses which contribute finite displacements will affect the aggregate content of T. Now the centres of these will lie either in the interval, or at a distance from its extremities of order $1/c$. As $1/c$ is small compared with T we shall practically be concerned only with the large number of pulses which lie almost entirely within T. We are, therefore, justified in regarding $1/c$ as the effective order of breadth of these strictly infinite pulses.

§ 29. If the pulses (supposed finite) are crowded, so that they overlap largely, we shall not find the characteristic spectrum unless the time-interval T which we are investigating is large compared with the breadth of a pulse. If T cuts into many pulses, but is not large compared with the breadth of each, we shall lose sight of individual pulses, it is true; but the energy function will be largely affected by the incomplete pulses. In other words, if we can shorten the exposure till it is comparable with the duration of a pulse, the spectrum observed will begin to show deviations from the normal spectrum as taken with a much larger exposure, or as observed by the eye.

§ 30. The extension to aggregates of pulses which are not all similar is obvious. Suppose for instance that we have a sequence of pulses of constant displacement, the lengths of the pulses varying, while at the same time the proportions of different lengths are given. The pulses of lengths between x and $x + dx$ are, say, $f(x)\, dx$, of the whole. They may be taken as equal pulses; suppose that they give an energy function $\phi^2(u, x)$. Then the whole energy of the mixture is

$$\int_0^\infty\int_0^\infty f(x)\phi^2(u,x)\, dx\, du.$$

Röntgen Rays and Ordinary Light.

§ 31. Professor THOMSON* explains Röntgen radiation by supposing it to consist of a succession of electro-magnetic pulses. Each pulse is practically a pulse of constant magnetic force, lasting for a short time. The thickness of a pulse is comparable with the diameters of the particles composing the cathode stream. Lord RAYLEIGH has pointed out† that these pulses may be regarded as simple waves of short wave-length. He did not explicitly consider the properties of a *succession* of pulses. Perhaps on account of this insufficiency of statement, Professor THOMSON‡ has not fully accepted the above-mentioned view. He has held that the Fourier analysis of a pulse has no physical meaning. Now this is a valid objection to the identification of the single pulse with ordinary light of any composition whatever. The different elements of the integral will possess definite phase-relation; they are in no sense independent.

On the other hand, it has been proved in the course of the present essay that the succession of pulses will actually be equivalent to a spectrum of definite composition. The Thomson pulses will certainly possess the property of being brief in comparison with the shortest observable interval of time; there will be a great number of them in such an interval; it follows that, as the instrument averages over small ranges of wave-length, phase properties will be lost; the processes of time- and wave-length averaging will efface all distinction between the succession of pulses and that mixture of light which is determined by the analysis of the single pulse.

§ 32. We proceed to consider the energy-distribution in the scale of wave-length.

We must express as a Fourier integral a function of x which is zero from $-\infty$ to $-d$; E from $-d$ to $+d$; zero from $+d$ to $+\infty$.

We find

$$\phi(x) = \frac{2E}{\pi}\int_0^\infty \frac{\sin ud}{u}\cos ux\, du.$$

* Professor THOMSON, 'Phil. Mag.,' February, 1898.

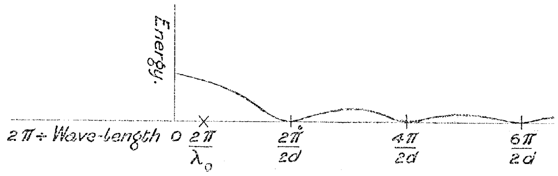
† Lord RAYLEIGH, 'Nature,' April 28, 1898.

‡ Professor J. J. THOMSON, 'Nature,' May 5, 1898.

Accordingly the distribution of energy in the spectrum given by the succession of such pulses is that shown in the curve

$$y = \frac{\sin^2 ud}{u^2}.$$

Fig. 1.—Energy Curve of Röntgen Rays.



If we take the particles of the cathode stream to be at least as great as molecules, $2d$, the thickness of pulse, is small compared with the wave-length of visible light (see THOMSON'S paper); d/λ_0 may be taken as $\frac{1}{1000}$, where λ_0 is the wave-length of yellow light, say. In the scale of fig. 1 $2\pi/\lambda_0$ is very near to 0. It appears that long waves have the greatest amplitude; practically the same amplitude is maintained onwards through the visible spectrum, and in fact till we approach to wave-lengths comparable with the diameter of the molecules.

§ 33. The measure with which we are concerned, however, is not amplitude, but integral energy through ranges of wave-length. Considering this, we see at once that the short waves are all-important. The total energy of the pulse is of order $E^2 d$. The energy contained in waves of length from infinity to λ_0 is of order $E^2 d. (d/\lambda_0)$. Remembering that the visible spectrum includes an octave, we may say, roughly, that $\frac{1}{1000}$ of the energy of the radiation will reside in the visible part of the spectrum; and, of the rest, practically the whole in waves of length comparable with the diameter of the molecules. It is noteworthy that waves of length equal to the thickness of the pulse or sub-multiples thereof will be excluded from the spectrum.

§ 34. Inequalities in the thickness of the pulses will slightly modify the features of the equivalent spectrum. Such inequalities will arise partly from the fact that different pulses arrive in slightly different directions; they come from different parts of the glass (an effect diminishing with distance). Furthermore, the particles are not stopped at a single impact in the molecules of the glass.

It also appears that, if Röntgen rays can be made powerful enough, they will affect the eye as ordinary white light.

§ 35. Professor THOMSON'S magnetic pulses are all negative. A mixture of negative and positive pulses will present the same features except in so far as the long-waves are concerned. If the negative and positive are present in equal quantities, the amplitude of the infinite wave will vanish.

§ 36. It is to be remarked that Professor THOMSON'S magnetic pulses differ in one important respect from the thin pulses by which Sir GEORGE STOKES* has sought to

* WILDE Lecture, 'Proc. Manchester Phil. Soc.,' 1897.

explain Röntgen rays. In the former, we have the magnetic force great and negative throughout the pulse; in the latter, positive and negative are to be so balanced that the force integrated through the thickness of the pulse shall vanish. On this property, together with that of the thinness of the pulses, Sir GEORGE STOKES* bases his proof that there will be no sensible diffraction.

The mathematical consequence of this property will be zero amplitude for the infinite wave-length. Practically this means that the energy in the visible spectrum is very much smaller than for the Thomson pulse. It will be of order $E^2 d \cdot (d/\lambda_0)^3$, instead of $E^2 d \cdot d/\lambda_0$; a proportion of 10^{-9} of the whole, instead of 10^{-3}. Now diffraction depends chiefly on waves whose lengths are of this order; very much shorter waves will not be diffracted, but will penetrate matter; and in any case would give much smaller diffraction patterns. Pulses of both the proposed forms will be sensibly free from diffractive properties; those of STOKES in a much higher degree than those of THOMSON.

Radiation of an Incandescent Gas.

§ 37. As an example of the composition of a large number of independent pulses of uniform type, we will take the case of radiation from an incandescent gas. We will suppose the mass of gas to be at a great distance, and to have no visible diameter; we shall thus be enabled to consider the radiation as composed of plane-waves travelling at right angles to their own wave fronts. Furthermore, the amount of gas is to be so small that the emission is not sensibly affected by absorption. The gas is to consist of molecules all having the same period of free vibration.

§ 38. The light received by the spectator will not be homogeneous. One reason for this is the Doppler effect.† The velocities of the molecules in the line of sight will alter the period of the light received. Another cause will doubtless be the altered vibrations of two molecules when very near to one another. This will perhaps become important at high pressures, but we will not further consider it at present.

§ 39. Lastly, we have to take into account the fact that the train of single waves emitted by each vibrating molecule is not infinite in length, but has a definite beginning and ending. The effect of this cause is investigated below. The Doppler effect is included in the same piece of analysis. We shall arrive at the remarkable result that the limiting width of the spectrum line when the pressure is indefinitely diminished is *less* by some 10 per cent. than the width calculated by Lord RAYLEIGH, who took into account nothing but the Doppler effect.

§ 40. The vibration of a molecule will be altered by collision with another. The velocity will also be altered, in both magnitude and direction. The vibration *received by the spectator* from this molecule will therefore be suddenly and fortuitously altered in period, amplitude and phase. The total radiation received

* WILDE Lecture, 'Proc. Manchester Phil. Soc.,' 1897.

† Lord RAYLEIGH, 'Phil. Mag.,' vol. 27, April, 1889.

from the whole gas will consist of a great number of finite trains superposed. We must consider those trains as practically independent. It is true that each individual train is connected in one respect, namely, instant of beginning or ending, with two others, the trains emitted by the same molecule before and after. But this element of regularity will be overwhelmed by the independence of the different molecules. All we have to do is, to find the Fourier integral equivalent to a finite train of waves, to find the distribution of energy in the scale of frequency, and to sum up the energy for all possible trains.

Fourier Analysis of a Train of m Complete Sine Waves.

§ 41. The general theorem is

$$\pi f(x) = \int_0^\infty \int_{-\infty}^{+\infty} \cos \omega(\lambda - x) f(\lambda) d\omega\, d\lambda$$

In the present case $f(x) = 0$, except from 0 to $2\pi m/\kappa$, within which limits $f(x) = \cos \kappa x$.

Thus

$$\pi f(x) = \int_0^\infty \int_0^{2\pi m/\kappa} \cos \omega(\lambda - x) \cos \kappa\lambda\, d\omega\, d\lambda$$

$$= \tfrac{1}{2}\int_0^\infty \int_0^{2\pi m/\kappa} \{\cos \overline{(\omega + \kappa)\lambda - \omega x} + \cos \overline{(\omega - \kappa)\lambda - \omega x}\} d\omega\, d\lambda$$

$$= \int_0^\infty d\omega \left\{ \frac{\cos \omega\left(x - \frac{\pi m}{\kappa}\right) \sin \frac{\pi m \omega}{\kappa}}{\omega + \kappa} + \frac{\cos \omega\left(x - \frac{\pi m}{\kappa}\right) \sin \frac{\pi m \omega}{\kappa}}{\omega - \kappa} \right\}$$

If we consider the quantities $\dfrac{\sin \frac{\pi m\omega}{\kappa}}{\omega + \kappa}$, $\dfrac{\sin \frac{\pi m\omega}{\kappa}}{\omega - \kappa}$, we see that the latter attains to a maximum value $\pi m/\kappa$ at $\omega = \kappa$, and that the former is small in comparison since m is generally a considerable number.

We shall be concerned only with the values of ω near to κ; accordingly the first term shall be neglected. We shall then have a distribution of energy,

$$\int_0^\infty d\omega \frac{\sin^2 \pi m\omega/\kappa}{(\omega - \kappa)^2},$$

neglecting numerical coefficients which do not alter the distribution.

Let $\omega - \kappa = 2\pi n$; then at a distance n from the maximum (n being reciprocal wave-length) we have energy proportional to

$$\frac{\sin^2 \frac{\pi m}{\kappa}(\omega - \kappa)}{n^2} = \frac{\sin^2 \pi r n}{n^2}$$

where r is the length of the train.

For this single train, then, we have energy falling off from a maximum at κ according to the law

$$\frac{\sin^2 \pi r n}{n^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.})$$

(the ordinary law for the diffraction pattern of an edge), where n is the distance from the brightest part measured in the scale of reciprocal wave-length.

Summation for all Molecules having Definite Velocities both Athwart and in Line of Sight.

§ 42. For these we have a definite position κ of maximum brightness, and definite resultant velocity v. We have to integrate for the different lengths of train.

Now TAIT* has shown that, of all atoms moving with velocity v, a fraction $e^{-f\rho}$ penetrates unchecked to distance ρ, where

$$f = 4\pi n s^2 \ . \ \frac{h^{\frac{3}{2}}}{\pi^{\frac{1}{2}}}\left(\frac{1}{4h^2 v}e^{-hv^2} + \overline{\frac{1}{4h^2v^2} + \frac{1}{2h}}\int_0^v e^{-hv^2}dv\right).$$

We may write this function of v as follows :—

$$f = \pi^{\frac{1}{2}} n s^2 \left[\frac{e^{-P^2}}{P} + \left(\frac{1}{P^2} + 2\right)\int_0^P e^{-P^2}dP\right]$$

$$= c^2 f(P) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii.}),$$

where $P = vh^{\frac{1}{2}}$,

$$f(P) = \frac{e^{-P^2}}{P} + \left(\frac{1}{P^2} + 2\right)\int_0^P e^{-P^2}dP,$$

$$c^2 = \pi^{\frac{1}{2}} n s^2,$$

n = number of atoms in unit volume,

s = diameter of atom.

From this we see that, of molecules moving with velocity v, a fraction

$$fe^{-f\rho}d\rho$$

have free paths between ρ and $\rho + d\rho$.

Now, such a molecule will emit an undisturbed train of waves of length between r and $r + dr$, where $r = \frac{V}{v}\rho$, and V is the velocity of light.

Hence, of all molecules moving with velocity v, a fraction $\frac{vf}{V}e^{\frac{-vfr}{V}}dr$, will give free paths between r and $r + dr$.

* TAIT, 'Edinb. Trans.,' vol. 33 p. 72.

Returning to the expression for the energy of a single train of length r (i.), we see that with the aggregates of molecules now under consideration (definite thwart and line-of-sight velocities) we have for n a proportion of energy

$$\frac{fv}{n^2 V} \int_0^\infty e^{-\frac{vf}{V} \cdot r} \sin^2 \pi n r \,.\, dr$$

$$= \frac{fv}{2n^2 V} \int_0^\infty e^{-\frac{vf}{V} r} (1 - \cos 2\pi n r)\, dr$$

$$= \frac{fv}{2n^2 V} \left| \left(-\frac{V}{vf} - \frac{-\frac{vf}{V}\cos 2\pi n r + 2\pi n \sin 2\pi n r}{\left(\frac{vf}{V}\right)^2 + 4\pi^2 n^2} \right) e^{-\frac{vfr}{V}} \right|_0^\infty$$

$$= \tfrac{1}{2} \frac{1}{n^2 + \left(\frac{vf}{V}\right)^2} \quad \text{(iii.)}$$

We next integrate for a definite velocity p in the line of sight, and all possible velocities q athwart.

The proportion of molecules with thwart velocities between q and $q + dq$ is $qe^{-hq^2}dq$. Hence, omitting the $\frac{1}{2}$ from (iii.) (it does not affect the distribution of energy), we have

$$\int_0^\infty qe^{-hq^2} dq \,.\, \frac{1}{n^2 + \left(\frac{vf}{V}\right)^2} \quad \text{(iv.)}$$

where $v^2 = p^2 + q^2$.

Lastly, we introduce all possible velocities in line of sight.

Here the Doppler effect enters; the mid-point of the spectrum (iv.) will be different for different p's. Let x be the distance from the centre of the final spectrum line (measured, as before, in reciprocal wave-lengths), we have

$$\int_{-\infty}^{+\infty} e^{-hp^2} \int_0^\infty qe^{-hq^2} \frac{1}{\left(x - \frac{p}{\lambda V}\right)^2 + \left(\frac{vf}{2\pi V}\right)^2} dp\, dq \,. \quad \text{(v.)}$$

This integral, regarded as a function of x, gives the distribution of light in the spectrum.

To make further progress, we will change the variables of integration from p and q to p and v, where $v^2 = p^2 + q^2$. We must remember that f is a function of v.

We have

$$\int_0^\infty e^{-hp^2}dp \int_p^\infty ve^{-hq^2}\frac{1}{\left(x-\frac{p}{\lambda V}\right)^2+\left(\frac{vf}{2\pi V}\right)^2}dv$$

$$+\int_{-\infty}^0 e^{-hp^2}dp \int_{-p}^\infty ve^{-hq^2}\frac{1}{\left(x+\frac{p}{\lambda V}\right)^2+\left(\frac{vf}{2\pi V}\right)^2}dv$$

$$=\int_0^\infty\int_p^\infty e^{-hv^2}\,.\,v\,dp\,dv\left\{\frac{1}{\left(x-\frac{p}{\lambda V}\right)^2+\left(\frac{vf}{2\pi V}\right)^2}+\frac{1}{\left(x+\frac{p}{\lambda V}\right)^2+\left(\frac{vf}{2\pi V}\right)^2}\right\},$$

or, changing the order of integration,

$$\int_0^\infty\int_0^v dv\,dp\,.\,ve^{-hv^2}\left\{\frac{1}{\left(x-\frac{p}{\lambda V}\right)^2+\left(\frac{vf}{2\pi V}\right)^2}+\frac{1}{\left(x+\frac{p}{\lambda V}\right)^2+\left(\frac{vf}{2\pi V}\right)^2}\right\}\quad .\quad .\;\text{(vi.)}$$

Visibility Curve.

§ 43. Professor MICHELSON* has shown that, although the breadths of elementary spectrum lines cannot in general be examined directly, yet the application of his interference method enables one to obtain much more detailed information. The light is made to interfere with itself, at a relative retardation u of the two half streams. Interference bands are produced and their "visibility" estimated for different values of u, the path-difference. From the visibility-curve thus constructed we can work backwards to the breadth of the spectrum line, and find out something about the distribution of light in this breadth.

MICHELSON has shown that, if $\phi(x)$ represent the intensity of light for position x in the spectrum, and

$$C=\int_{-\infty}^{+\infty}\phi(x)\cos 2\pi ux\,dx$$

$$S=\int_{-\infty}^{+\infty}\phi(x)\sin 2\pi ux\,dx$$

$$Q=\int_{-\infty}^{+\infty}\phi(x)dx$$

and

$$V^2=\frac{C^2+S^2}{Q^2},$$

then V is the visibility-function, in terms of u, the path-difference.

* MICHELSON, 'Phil. Mag.,' vols. 31 and 34.

In the present case $S = 0$, and

$$C = \int_{-\infty}^{+\infty} \int_0^{\infty} \int_0^{v} \cos 2\pi ux \, dx \, . \, e^{-hv^2} . \, v \, dv \, dp \Sigma \frac{1}{\left(x \pm \frac{p}{\lambda V}\right)^2 + \left(\frac{vf}{2\pi V}\right)^2}$$

Now

$$\int_{-\infty}^{+\infty} \cos 2\pi ux \frac{dx}{\left(x - \frac{p}{\lambda V}\right)^2 + \left(\frac{vf}{2\pi V}\right)^2}$$

$$= \cos \frac{2\pi up}{\lambda V} \int_{-\infty}^{+\infty} \frac{\cos 2\pi ux \, dx}{x^2 + \left(\frac{vf}{2\pi V}\right)^2} = \cos \frac{2\pi up}{\lambda V} . \frac{2\pi^2 V}{vf} . e^{-\frac{vfu}{V}}.$$

We may omit certain constant factors, and write

$$C = \int_0^{\infty} \int_0^{v} \cos \frac{2\pi up}{\lambda V} . \frac{1}{vf} . e^{-\frac{vfu}{V}} e^{-hv^2} . \, v \, dv \, dp$$

$$= \frac{\lambda V}{2\pi u} \int_0^{\infty} \sin \frac{2\pi uv}{\lambda V} . \frac{du}{f} . e^{-\left(hv^2 + \frac{vfu}{V}\right)}.$$

We may still further simplify this by introducing the notation of page 347. Omitting unnecessary factors,

$$C = \frac{1}{u} \int_0^{\infty} \sin \frac{u\kappa P}{V h^{\frac{1}{2}}} . \frac{dP}{f(P)} e^{-P^2} e^{-\frac{\pi^{\frac{1}{2}} n s^2}{V h^{\frac{1}{2}}} . \, uPf(P)}$$

$$= \frac{1}{u} \int_0^{\infty} \sin 2buP . \frac{dP}{f(P)} . e^{-P^2 - a^2 uPf(P)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii.)},$$

where

$$2b = \frac{\kappa}{V h^{\frac{1}{2}}}$$

$$a^2 = \frac{\pi^{\frac{1}{2}} n s^2}{V h^{\frac{1}{2}}}$$

$$P = v h^{\frac{1}{2}}$$

$$f(P) = \frac{e^{-P^2}}{P} + \left(\frac{1}{P^2} + 2\right) \int_0^P e^{-P^2} dP.$$

To deduce V (the visibility function) from C, all we have to do is to put $u = 0$ in C, and divide C by the quantity thus formed.

Limiting Case of Zero Pressure.

§ 44. If the pressure is very small, n, the number of molecules in unit volume becomes small, and with it a^2. We are thus reduced to

$$C_0 = \frac{1}{u} \int_0^{\infty} \sin 2buP . \frac{e^{-P^2} dP}{f(P)}$$

* TAIT, 'Edinb. Trans.,' vol. 33, p. 95.

§ 45. This expression as it stands is quite intractable. We are enabled, however, to make progress by means of tables which TAIT* gives in his paper on the kinetic theory. Our subject of integration is

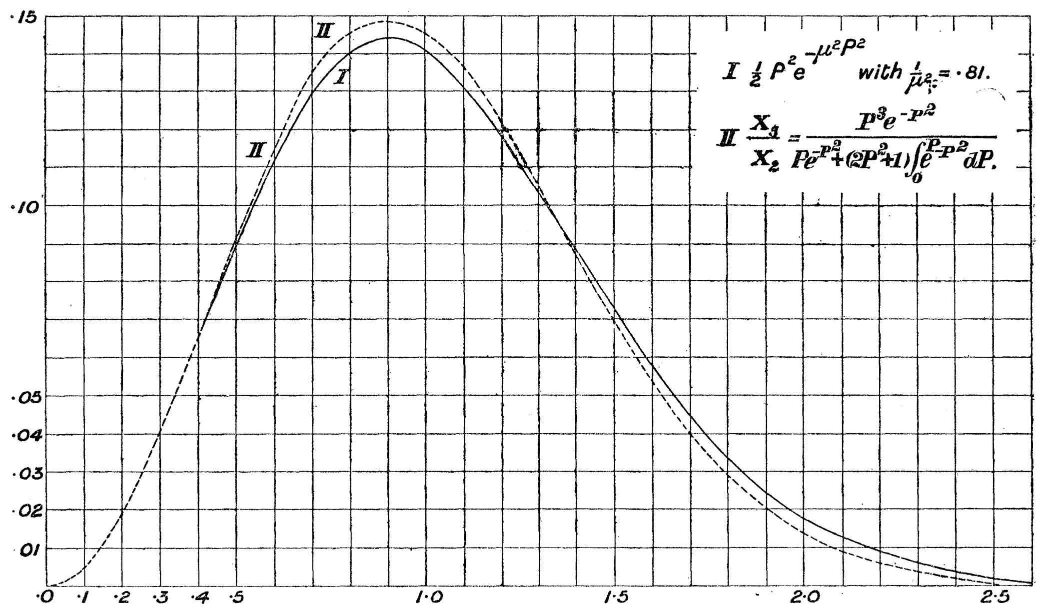
$$\frac{e^{-P^2}\sin 2buP}{f(P)} = \frac{\sin 2buP}{P} \cdot \frac{P^3 e^{-P^2}}{Pe^{-P^2} + (2P^2+1)\int_0^P e^{-P^2}dP}$$

$$= \frac{\sin 2buP}{P} \cdot \frac{X_3}{X_2} \text{ in TAIT's notation.}$$

On marking out the graph of X_3/X_2 by means of the values in column 6 of TAIT's table, it becomes obvious that the general outline of the function is very near to that of

$$\tfrac{1}{2}P^2 e^{-\mu^2P^2}.$$

Fig. 2.



To make the agreement as good as possible we choose μ so that the two curves may attain their maximum for the same value of P.

Now the latter function has its maximum at $P = \frac{1}{\mu}$; graphically we see that we must choose $\frac{1}{\mu} = \cdot 9025$ (about); $\frac{1}{\mu^2} = \cdot 81$ . . .

In the accompanying figure, I. (continuous curve) is X_3/X_2; II. (dotted curve) is $\frac{1}{2}P^2e^{-P^2/\cdot 81}$. The agreement is so good that the error in the integral through using II. instead of I. will be only about 1 per cent. But this is of the order of the errors in the observed visibilities, with which we propose to compare our results.

Making this substitution, we have

$$C_0 = \frac{1}{2u}\int_0^\infty \frac{\sin 2buP}{P} \cdot P^2 e^{-\mu^2 P^2} dP = \frac{b\pi^{\frac{1}{2}}}{4\mu^3} e^{-\frac{b^2u^2}{\mu^2}}$$

$$\therefore \quad V_0 = e^{-\frac{b^2u^2}{\mu^2}} = e^{-\frac{\kappa^2u^2}{4\mu^2V^2\lambda}} \quad . \quad . \quad . \quad . \quad . \quad (\text{viii.})$$

§ 46. The result is now in a form which we can interpret. Lord RAYLEIGH* has worked out the width of the spectrum lines, taking into account the Doppler effect alone. The visibility function deduced from his work in accordance with MICHELSON's definition of "visibility"† is

$$e^{-\kappa^2u/4V^2h} \quad . \quad . \quad . \quad . \quad . \quad (\text{ix.})$$

The two functions (viii.) and (ix.) are of the same form, and differ only in the presence of μ^2 in (viii.). Now MICHELSON's experiments gave visibility-curves agreeing in general character with (ix.); they would therefore agree equally well with the function (viii.) which has been found above.

Numerical Estimates.

§ 47. MICHELSON has investigated the lines of several gases with his interferometer. He compares the "half-widths" (value of u for which visibility is $\frac{1}{2}$ of maximum) of the visibility-curves with the values deduced by Lord RAYLEIGH from DOPPLER's principle. Unfortunately MICHELSON misquotes RAYLEIGH's result, and has dropped a 2; in RAYLEIGH's formula the path-difference is 2Δ; MICHELSON has taken it to be Δ. The last column of figures in page 294 of MICHELSON's paper‡ should be all doubled. This would give the observed "half-widths" in every case much less than the calculated values. Furthermore, using the function which has been obtained in the present paper,

$$V = 2^{-\frac{\kappa^2u^2}{4l\mu^2V^2h}}$$

where l is the Napierian log. of 2.

Hence for half-width

$$\frac{\kappa u}{2\mu\sqrt{l}Vh^{\frac{1}{2}}} = 1$$

$$\therefore \quad u = \frac{2\mu V\lambda^{\frac{1}{2}}\sqrt{l}}{\kappa}.$$

But $v = \frac{2}{\pi^{\frac{1}{2}}h^{\frac{1}{2}}}$; v being the average velocity of the molecules.

Hence $$u = \lambda \cdot \frac{2\mu\sqrt{l}}{\pi^{\frac{3}{2}}} \cdot \frac{V}{v}.$$

$$\therefore \quad u/\lambda = \cdot 33V/v, \text{ instead of MICHELSON's } \cdot 15V/v.$$

* 'Phil. Mag.,' vol. 27, p. 304, 1889. † MICHELSON, 'Phil. Mag.,' vols. 31 and 34.

‡ 'Phil. Mag.,' vol. 34, 1892.

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Lord RAYLEIGH's expression for the half-width is

$$u = \lambda \,.\, \frac{2\sqrt{l}}{\pi^{\frac{3}{2}}} \,.\, \frac{V}{v}.$$

Now $\mu = 10/9$; hence the theory developed in the present paper necessitates a further addition of about 10 per cent. to the values calculated from Lord RAYLEIGH's formula. But the effect of this is quite obscured by the above-mentioned necessity of doubling the figures which MICHELSON deduces from theory. The conclusion of his paper should be that Lord RAYLEIGH's theory accounts for *a certain fraction of the observed widths of spectral lines.* The fraction varies from $\frac{2}{3}$ to $\frac{1}{4}$ for the different substances examined.

If the distribution of energy in the spectral line be given by

$$\phi(x) = 2^{-x^2/\delta_0^2} = e^{-\frac{lx^2}{\delta_0^2}},$$

then
$$C = \int \phi(x) \cos 2\pi ux \, dx = \frac{\pi^{\frac{1}{2}} \delta_0}{\sqrt{l}} \,.\, e^{-\frac{\pi^2 u^2 \delta_0^2}{l}}$$

and
$$V = e^{-\frac{\pi^2 u^2 \delta^2}{l}} = 2^{-\frac{\pi^2 u^2 \delta_0^2}{l^2}}.$$

If we construct the curve representing the energy of the spectrum on a scale of reciprocal wave-lengths, the "half-width" in this curve will be δ_0. Hence the "half-widths" in the energy-curve and the visibility-curve are connected by the relation

$$n_0 = \frac{l}{\pi} \,.\, \frac{1}{\delta_0}.$$

§ 48. Our result that 10 per cent. ought to be added to the visibility half-width means that the theoretical width of the spectrum line should be diminished by a similar percentage.

It might have been expected that, with pressure small and collisions comparatively few, the modifying effect of the curtailment of free trains would disappear, the free paths being now on the average long. We might have expected that the formula for the width of the lines would converge to that given by Lord RAYLEIGH, in which the Doppler effect alone is considered. If the above reasoning is valid, there is no such convergence of the two theories when the pressure is indefinitely reduced; the results derived from them differing by some 10 per cent. It is noteworthy that the present theory leads us to expect narrower lines than does Lord RAYLEIGH. This result is certainly paradoxical, and calls for further justification.

The modification in theory has been to substitute for mathematically homogeneous light proceeding from each molecule, a radiation giving a certain continuous spectrum. We could hardly have foreseen without a complete analysis that, for zero pressure, the integrated effect of all these spectra gives an intensity curve for the total spectral line *steeper* than before.

§ 49. The following considerations may tend to remove the doubts which this result may arouse.

The whole set of molecules with given velocities, thwart and in line-of-sight, and with given length of free path, will emit light of a certain spectral composition, namely, that given by the function $\frac{\sin^2 \pi rn}{n^2}$ (p. 346), r being the length of the train of waves emitted during the single free path. In theory, this will give a set of lines in the spectrum of the same pattern as the diffraction lines of a straight edge. Other sets of molecules with other velocities and path-lengths will give other sets of lines; the whole aggregate of lines overlapping and compounding to give such spectrum lines as actually exist, and were measured by MICHELSON.

§ 50. Now, by lowering the pressure of the gas, we may lengthen the average free path, and the average trains of waves emitted in a single flight, this lengthening being theoretically without limit. The effect of this will be to narrow the curve $\frac{\sin^2 \pi rn}{n^2}$, also without limit. We are tempted, when this happens, to substitute for the aggregate of curves $\frac{\sin^2 \pi rn}{n^2}$ the aggregate of their maximum ordinates. If we do this, and also assume that the average length of wave-train is the same for all different velocities in the gas, we shall, in effect, be following Lord RAYLEIGH'S procedure, and we shall obtain his expression for the width of the resultant line.

§ 51. On closer examination, it will be obvious that the molecules moving with greater velocities emit, on the average, *shorter* wave-trains. For, given the velocity, the mean free path is $1/f$ (p. 347), a function of v; while the corresponding train of waves has length V/vf. It is not difficult to verify that this function of n diminishes as v increases.

Now, the molecules with greater velocities in the line of sight have, on the average, greater resultant velocities. These, therefore, give shorter trains of waves, and smaller ordinates in the energy curve (the maximum of $\frac{\sin^2 \pi rn}{n^2}$ is $\pi^2 r^2$, for $n = 0$). But these molecules will provide light that goes towards the edges of the spectrum line. The energy-curve will accordingly be steeper, and the line narrower than would follow from the assumption that the mean free path is the same for two groups of molecules having two different velocities.

§ 52. Furthermore, it is not allowable to substitute for the component curves their maximum ordinates, however steep and narrow these curves may become. The maximum of $\frac{\sin^2 \pi rn}{n^2}$ is $\pi^2 r^2$. If we substitute these maximum ordinates and then form an energy-curve by summing them all into a smooth curve, each will contribute energy proportional to r^2. But, in reality, the total energy connected with $\frac{\sin^2 \pi rn}{n^2}$ is $\int_{-\infty}^{+\infty} \frac{\sin^2 \pi rn^2}{n^2} dn = \pi^2 r$. Thus, trains of length r ought to contribute total energy proportional to r, not to r^2. In fact, the taller the component curves are, the narrower they are. This consideration is overlooked if we allow ourselves to substitute the maxima of the curves for the curves themselves when these become narrow.

§ 53. This latter source of error, by which r^2 is substituted for r, will not affect the shape of the resultant energy-curve if the average length of train is taken to be independent of the velocity. But we have seen that greater velocities give shorter trains. And this error tends in the same direction as the other; for it gives too great prominence to the longer trains, *i.e.*, to the smaller velocities, which velocities send light to the middle of the spectrum line. Hence the effect of the error is to make the resultant curve too steep.

Accordingly, the only accurate way of investigating the limiting width for zero pressure is to form the general energy function as on p. 349, and then to proceed to the limit by diminishing the number of molecules in unit volume.

Effect of Damping in the Widths of Spectrum Lines.

§ 54. It has been urged by LOMMEL* that, whereas the vibrations of an atom are undoubtedly damped by radiation, the light emitted by a simple gas will be to some extent continuous. The same idea has been recently developed by JAUMANN.†

Both of these writers rely on a Fourier analysis of the vibration

$$e^{-\kappa t} \sin (pt + \psi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (x.)$$

Their results have not been accepted generally; as has been pointed out in the course of the present paper, their procedure will have no physical meaning for the single train of waves with which they deal.

But we have also seen that a vast aggregate of independent emissions of this form will really give the result which LOMMEL suggests.

§ 55. The trains of waves emitted by the molecules of a gas cannot strictly be represented by (x.). As a matter of fact, such a motion as (x.) will never be allowed to go on indefinitely; it will always be checked at a certain stage by a new collision. Nevertheless, if the radiation is so rapid that the vibration of a molecule has generally become insignificant before the next collision occurs, we shall not be making a serious error by allowing (x.) to represent the train of waves. We will proceed to investigate the effect on the assumption that the damping is so rapid as to allow this procedure.

§ 56. Both LOMMEL and JAUMANN make an erroneous application of FOURIER's theorem. The analysis should be as follows :—

* LOMMEL, 'Wied. Ann.,' vol. 3, p. 251, 1878.

† JAUMANN, 'Wied. Ann.,' vols. 53 and 54.

$$F(t) = 0 \text{ for } t = -\infty \text{ to } t = 0$$

$$\text{and} \quad F(t) = e^{-\kappa t}\cos pt \text{ for } t = 0 \text{ to } t = +\infty.$$

(The phase ψ will not make any important difference in the energy-function.)

$$\pi F(t) = \int_0^\infty \int_0^\infty e^{-\kappa\beta}\cos p\beta \cos u(\beta - t)du\,d\beta$$

$$= \tfrac{1}{2}\int_0^\infty du \int_0^\infty e^{-\kappa\beta}[\cos(\overline{p+u}\beta - ut) + \cos(\overline{p-u}\beta + ut)]d\beta$$

$$= \tfrac{1}{2}\int_0^\infty du\left\{\frac{\kappa\cos ut + \overline{p+u}\sin ut}{\kappa^2 + (p+u)^2} + \frac{\kappa\cos ut - \overline{p-u}\sin ut}{\kappa^2 + (p-u)^2}\right\}.$$

Now κ/p is small since the damping is gradual; accordingly, both $\frac{1}{\kappa^2 + (p+u)^2}$ and $\frac{1}{\kappa^2 + (p-u)^2}$ will be of order $1/p^2$ unless p is near to u. In that case, the latter of the two expressions will attain to the order $1/\kappa$. We are justified in approximating to the extent of neglecting the former expression.

§ 57. The energy of the train of waves will depend on

$$\int_0^\infty \frac{du}{\kappa^2 + (p-u)^2}.$$

This function will define the spectrum to which a vast concourse of such damped trains is equivalent.

Now this will be a widened line in the spectrum. The "half-width" will be of order κ in frequency. The half-widths which MICHELSON has observed for irresoluble lines are of order

$$10^{-5} \times p.$$

If κ is of this order, $\frac{\kappa \times 10^5}{p}$ is finite.

Now $10^5/p$ is comparable with the time of 10^5 vibrations. Again, there are on the average 10^5 vibrations in the free path. In time t the vibrations are reduced in the ratio $e^{-\kappa t} : 1$; if κt is finite, the reduction of the energy may be very noticeable.

§ 58. We are thus led to conclude that the vibrations of molecules may be very considerably damped in the course of their free paths and yet no widening of lines be produced beyond what is actually observed. It will be remembered that the kinetic theory, without damping, gave widths varying from one-quarter to two-thirds of the widths observed. It seems not impossible that, for small densities, the residual width is to be ascribed to radiative damping. When the density becomes considerable, the mutual effect of neighbouring molecules will doubtless become so important as to obscure both damping and Doppler effect

Character of the Æther Motions in nearly Homogeneous Light.

§ 59. It has been thought that the possibility of producing a large number of interference fringes from white light is an indication of a certain regularity in the æther motion corresponding to such light. This view has been abundantly refuted by GOUY, RAYLEIGH, and SCHUSTER.* The fringes cannot be produced without the use of a spectral apparatus; and the number of the fringes is an index, not of the regularity of the white light, but of the resolving power of the spectroscope.

A large number of fringes can also be produced without a spectroscope, by using radiations which naturally possess a high degree of homogeneity. The number of these fringes is a test of the homogeneity; and in this case, it is also a test of the regularity of the æther motion.

§ 60. We have justified the representation of light by a Fourier integral of the form

$$\int_0^\infty \mathrm{R} \cos (ut + \psi) du,$$

where R, ψ are functions of u.

It can be shown that, for light of long duration, ψ will fluctuate rapidly in terms of u.

For approximately homogeneous light of mean frequency p, we will use the notation

$$\int \mathrm{R} \cos (\overline{p + u}t + \psi) du.$$

In this expression R will be insignificant, except for values of u small compared with p.

The above integral can be written

$$\cos pt \int \mathrm{R} \cos (ut + \psi) du - \sin pt \int \mathrm{R} \sin (ut + \psi) du.$$

The ranges of integration will practically be confined to a region on either side of zero, small compared with p.

Each of the integrals

$$\int \mathrm{R} \cos (ut + \psi) du, \qquad \int \mathrm{R} \sin (ut + \psi) du$$

is a function of t, whose variations are slow compared with those of $\cos pt$. The expression $\cos pt \int \mathrm{R} \cos (ut + \psi)\, du$ may be taken to denote a simple vibration of period $2\pi/p$, whose amplitude varies slowly. The zeros of the expression will be, effectively, the zeros of $\cos pt$. Similar statements will apply to

$$\sin pt \int \mathrm{R} \sin (ut + \psi) du.$$

* GOUY, 'J. de Ph.,' series 2, vol. 5, p. 354; RAYLEIGH, Art. "Wave Theory," 'Encycl. Brit.'; 'Phil. Mag.,' vol. 27, p. 460, 1889; SCHUSTER, 'Phil. Mag., vcl. 34, p. 509, 1894

Finally, the motion which we have analysed as

$$\cos pt \int R \cos (ut + \psi) du - \sin pt \int R \sin (ut + \psi) du$$

may be described as a simple vibration whose amplitude *and phase* vary slowly. We might equally well say that the amplitude and period vary slowly; the latter within narrow limits. In passing it is interesting to note that all the effects of white light would be produced by an approximately simple vibration whose period varies slowly, but within wide limits; the whole range of variation being traversed a great number of times in the course of the shortest observable interval of time The great gap between 10^{-15} second (the period of vibration) and 10^{-7} second (the shortest observable time) will give room for a rate of variation small compared with the one measure, and great compared with the other.

§ 61. Returning to the nearly homogeneous light, let $\pm s$ be the effective range of integration in

$$\int R \cos (ut + \psi) du, \qquad \int R \sin (ut + \psi) du.$$

This of course means that R becomes small outside the limits $\pm s$. The width of the spectrum line will be of order s. Now the two integrals just quoted will give irregularly sinuous time-curves, the average extent of a sinuosity being of order $1/s$. Thus, the varying simple vibration

$$\int_{p-s}^{p+s} R \cos (ut + \psi) du$$

will have entirely changed in amplitude and phase after a time of order $1/s$. It will therefore be impossible to produce sensible interference with time-differences of more than $1/s$. This is another aspect of the fact that for lines of width s (measured in frequency), the maximum path-difference for interference is of order $1/s$.

§ 62. We can look at the same matter from yet another point of view. We may go back to the composition of the radiation from a gas. This we have seen to be built up of finite trains of waves. For the moment, let us omit the Doppler effect and take all the trains to be of the same period. In an interval during which only a small proportion of molecules collide, the amplitude and phase of the composite vibrations is but little altered. But after an interval comparable with the mean free time of molecules, most of the molecules will have obtained new and independent vibrations; the composite motion will be entirely altered in amplitude and phase. For the discharge tubes used by MICHELSON this time is of order 10^5 periods.

Again, isolating the Doppler effect, we deal with the superposition of infinite simple trains. We have already seen that the composite vibration will have a materially altered amplitude and phase after a time equal to the reciprocal of the range of frequency in the component trains. For MICHELSON'S experiments it happens that this time is again some 10^5 periods.

In the actual radiation the two causes coexist; the joint effect is of the same order; the radiation will attain independence after every 10^5 periods. In accordance with this fact, we have interference up to path-difference of some 10^5 wave-lengths; while the width of the lines corresponds to a fraction $1/10^5$ of the frequency of the light.

Effect of a Natural Light on a Vibration.

§ 63. We have already proved that the complete solution of

$$\ddot{x} + 2\kappa\dot{x} + p^2x = f(t) = \int_0^\infty \mathrm{R}\cos(ut + \psi)du$$

is
$$x = \int_0^\infty \frac{\mathrm{R}du}{(p^2 - u^2)^2 + 4\kappa^2u^2}\{(p^2 - u^2)\cos(ut + \psi) + 2\kappa u\sin(ut + \psi)\},$$

$f(t)$ being such a function of time as can actually represent a natural radiation (*see* footnote, page 336).

Now we have already pointed out the hypothetical character of this treatment of the molecule as a simple vibrator. Nevertheless the method has a historical interest, and may be regarded as a foreshadowing of the truth; it may be worth while to sketch the result of applying our analysis.

The composition of the exciting light is

$$\int_0^\infty \mathrm{R}^2du,$$

of the light emitted by the molecule

$$\int_0^\infty \frac{\mathrm{R}^2du}{(p^2 - u^2)^2 + 4\kappa^2u^2},$$

while the rate of absorption is dependent on

$$\int_0^\infty \frac{2\kappa u^2\mathrm{R}^2du}{(p^2 - u^2)^2 + 4\kappa^2u^2}.$$

We will take the light to be constant. Its effect on the vibrator is seen to depend entirely upon its spectrum.

§ 64. This is equally true for nearly homogeneous light. The effect of the irregularities in the light may be deduced from the observed widening of the spectrum line.

Now SELLMEIER treated of this problem in the paper which laid the foundations of the modern theory of dispersion.* He recognised that no natural radiation is a perfect train of simple waves, and he investigated the effect of the irregularities upon a vibrator. The period of the light was to differ from that of the natural vibrations of the molecule. He came to the conclusion that the irregularities in the light would

* SELLMEIER, 'Pogg. Ann.,' 1872, vol. 145, p. 520.

not arouse the natural vibrations of the molecule. He held that the motion of the molecule would be mainly in the mean period of the incident light. This conclusion we take to be erroneous.

§ 65. SELLMEIER does not adopt the Fourier method; he builds up the incident light by means of a great number of finite superimposed trains of simple waves, all having the same phase and period, but of amplitudes, durations, and positions so adjusted as to give the actual fluctuation of amplitude which is present in the resultant motion. In passing, it may be noticed that this arrangement will give no irregularity of *phase* in the light motion; whereas we have shown that such irregularity will generally be present. The defect may be remedied by removing the condition that the component trains shall be of the same phase; but this consideration will not alter SELLMEIER's reasoning in any essential.*

SELLMEIER supposes the motion of the vibrator to be free from all damping. We will show that in this case his conclusion ought to be that the natural periods of the vibrator become continually more and more prominent without limit.

§ 66. His reasoning is as follows. Each new train of waves, as it strikes the vibrator, arouses :—

i. a vibration in the period of the incident light (forced),
ii. a vibration in the period proper to the vibrator (natural).

The forced vibration (i) will be of the same phase as the exciting train, and of amplitude proportional to that of the said train. The natural vibration will generally have an amplitude of the same order; its phase will, however, be different.

§ 67. The motion of the vibrator at any time results from the superposition of all the vibrations previously started. It will be partly forced and partly "natural." The forced vibrations at any instant will clearly differ from the incident motion at that instant by a numerical factor only.

The natural vibration, on the other hand, is compounded of members, whose phases are practically fortuitous. This is easily seen as follows. The phase of natural vibration aroused by the beginning of a new train depends upon the point of time at which this event takes place. The difference of a fraction of a period in the position of this point of time produces a finite change in the phase of the natural vibration aroused. But, in building up the slowly varying vibration by means of simple trains, the instant at which each start is, to a few periods, immaterial.

§ 68. SELLMEIER concludes from this that the natural vibration will be insignificant compared with the forced vibration. This mistake arises from the fact that

* It will be convenient to get rid of the endings of these trains. This we may do by supposing that a train, once started, is unending; and by introducing a train of equal amplitude and opposite phase, whose beginning is so fixed as to extinguish the former train at the right moment. By this arrangement we shall only have to deal with beginnings.

he really considers the effect of only a single waxing and waning of the incident light. It is true that, on the average, the amplitude of the sum of a large number of vectors of random phase is small compared with the sum of the amplitudes. At the same time the energy is, on the average, equal to the sum of the component energies. In the present case the right deduction is, that the energy of the natural vibration will vary with the number of the component vibrations; in other words, will vary as the time elapsed since the light began to act. It will become greater without limit. It is easily seen from SELLMEIER'S analysis that there is no tendency for the natural vibrations excited by successive fluctuation to counteract one another. As regards the forced vibrations, on the other hand, the phases are, so to speak, arranged so that there shall be no accumulation of energy.

§ 69. For a frictionless vibrator, then, common homogeneous light will give a continually-increasing motion in the natural mode; this feature being entirely due to the irregularities.

But if there were a friction, however small, the motion would be prevented from mounting up indefinitely. The vibrations started by the component trains would not persist; in fact we may expect to find that the natural vibration settles down to a definite state, depending on the damping and the nature of the irregularities.

§ 70. The whole of this matter becomes quite simple on the application of Fourier. Let us first try to solve

$$\ddot{x} + p^2x = \int_0^\infty \mathrm{R}\cos(ut + \psi)du.$$

We are tempted to take for solution

$$\int_0^\infty \frac{\mathrm{R}\cos(ut + \psi)}{p^2 - u^2}\,du.$$

This expression, however, has no definite value. The integrand involves an infinity at $u = p$; furthermore, the infinity is of such a nature that the integral

$$\int_0^{p-\epsilon} + \int_{p+\epsilon'}^\infty$$

depends upon ϵ/ϵ'.

The above integral, in fact, is not a solution of the equation. We are forced to include a frictional term in the equation. But this corresponds to the actual properties of the vibrator; we have shown that, but for damping, the natural vibration would continually increase; a state of things unknown among the observed effects of light.

§ 71. On page 336 it has been shown that light of composition $\int \mathrm{R}^2du$ will excite vibration of composition

$$\int \frac{\mathrm{R}^2du}{(p^2 - u^2)^2 + 4\kappa^2u^2}.$$

For nearly homogeneous light, of period $\frac{2\pi}{q}$, R is small if $\frac{q-u}{q}$ is finite. The integrand is therefore unimportant, except for values of u

i. near to q, where we may neglect κ, and use $\frac{1}{p^2-q^2}\int R^2 du$,

ii. near to p, where the emission is practically $\frac{1}{4p^2}\int\frac{R^2 du}{\kappa^2+(p-u)^2}$.

If the light emitted from the vibrator is analysed by a spectroscope, theoretically a spectrum of two lines should be revealed; the lines being at frequencies p and q. Let d be the half-width of the bright line in the incident light; then d will also be the half-width of the q line in the emitted light.

The total intensity of the q line in the emitted light is therefore of order $R_q^2 d/p^4$.

The half-width of the p line will be κ; the total intensity of the p line is of order $R_p^2/\kappa p^2$.

The ratio of these intensities is

$$\frac{Eq}{Ep} = \frac{R_q{}^2}{R_p{}^2} \cdot \frac{\kappa}{p} \cdot \frac{d}{p}.$$

Now R_q^2/R_p^2 is great, the incident light at p being by hypothesis invisible. On the other hand κ/p and d/p are both small. It therefore appears that, so far as the theory of the vibrator carries us, the natural vibration may be as prominent as the forced vibrator; the natural vibration varying inversely as the index of damping. Whether or no it is strong enough to be visible, depends upon the spectrum of the incident light and the constants of the vibrator.

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